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RAN-2203000206020031

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Third Year B. Sc. (Sem. - VI) Examination September - 2025

Mathematics (Paper - MTH-601) Ring Theory

Time: 2 Hours] **[Total Marks: 50**

સૂચના : / Instructions

(1) **નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.**
Fill up strictly the details of ☛ signs on your answer book
Name of the Examination:
☛ **Third Year B. Sc. (Sem. - VI)**
Name of the Subject :
☛ **Mathematics (Paper - MTH-601) Ring Theory**
Subject Code No.: **2203000206020031**

Seat No.:

Student's Signature

(2) All questions are compulsory.
(3) Figures to the right indicate marks of the questions.
(4) Follow usual notations.

Q. 1. Answer the following as directed: (Any FIVE) **(10)**

1. Does the ring $(R = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}\}, +_{20}, \times_{20})$ have a *Unit* element? Justify your answer.
2. In a ring R; prove that $a \cdot (-b) = -(a \cdot b)$; for all a, b in R.
3. Let $\phi : R \to R'$ be a homomorphism of a ring R into a ring R'. Then prove that $\phi(0) = 0'$.
4. If U is an ideal of a ring R with a *unit* element 1 and $1 \in U$, then $U = R$.
5. Express a greatest common divisor $\overline{4}$ of $\overline{2}$ and $\overline{5}$ as a linear combination of elements $\overline{2}$ and $\overline{5}$ in the Euclidean ring J_7 ; of integers modulo 7.
6. Show that 3 is not a greatest common divisor of 6 and 12 in the commutative ring $3\mathbb{Z}$.
7. Let a, b be elements in a Euclidean ring R. If $(a, b) = d_1$ and $(a, b) = d_2$, then prove that d_1 and d_2 are *associates*.
8. Define Prime Element. Justify: 2, 3, 5, 7, 11, 13, 17, 19, 23, ... all are prime elements in the Euclidean ring $\mathbb{R}$; of all real numbers.

Q. 2. **Answer any Two of the following:** **(10)**

a. Define Field. Answer as directed:

   i. Find the *Unit* element in the commutative ring $(R = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}\}, +_{20}, \times_{20})$

   ii. Is $(R = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}\}, +_{20}, \times_{20})$ integral domain? Justify your answer.

b. Prove that the commutative ring D is an integral domain if and only if $a, b, c \in D$ with $a \neq 0$; the relation $a \cdot b = a \cdot c \Rightarrow b = c$ holds in D.

c. Give an example of Boolean ring. Prove that every Boolean ring is commutative.

Q. 3. **Answer any Two of the following:** **(10)**

a. **Answer as directed:**

   i. Let $\phi : R \to R \mid U$ be a homomorphism of a ring R onto a Quotient ring $R \mid U$; defined by $\phi(a) = a + U$; for every $a \in R$. Find the Kernel $I(\phi)$ of a homomorphism ϕ.

   ii. Let $\phi : R \to R'$ be a homomorphism of a ring R onto a ring R'. If R is a ring with a *unit* element 1, then prove that R' is also a ring with a *unit* element $\phi(1)$.

b. Let $\phi : R \to R'$ be a homomorphism of a ring R into a ring R'. Then prove that ϕ is an isomorphism if and only if $I(\phi) = (0)$; where $I(\phi)$ is the Kernel of a homomorphism ϕ.

c. Which is the maximal ideal of a field F ? If F is a field, then prove that its only ideals are (0) and F itself.

Q. 4. **Answer any Two of the following:** **(10)**

a. **Answer as directed:**

   i. Find the Unit element in the commutative ring $(R = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}, +_{10}, \times_{10})$.

   ii. Show that $(R = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}, +_{10}, \times_{10})$ is the Euclidean ring.

b. Find all *Units* in the commutative ring ($R = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}\}, +_{15}, \times_{15}$); with the unit element $\overline{6}$. If; for a, b in an integral domain R with a *unit* element; both $a \mid b$ and $b \mid a$ hold true, then prove that $a = u\,b$; where u is *unit* in R.

c. Let $a \neq 0$, $b \neq 0$ in a Euclidean Ring R. Prove that:

   i. If b is *unit* in R, then $d(a) = d(ab)$.

   ii. If a is *unit* in R, then $d(a) = d(1)$.

Q. 5. Answer any Two of the following: (10)

a. For elements a, b, c and π in a Euclidean ring R, prove that:

   i. If $a \mid bc$ and $(a, b) = 1$, then $a \mid c$.

   ii. If π is a prime element in R and $\pi \mid ab$, then $\pi \mid a$ or $\pi \mid b$.

b. Prove that the relation of "*associates*" in a commutative ring R with a *unit* element is an equivalence relation on R.

c. Let R be a Euclidean ring and $a \neq 0$, $b \neq 0$ in R. If b is not *unit* in R, then prove that $d(a) < d(ab)$.